Simultaneous-Move Games: Mixed Strategies

Games Of Strategy
Chapter 7
Dixit, Skeath, and Reiley
Terms to Know

- Expected Payoff
- Opponent’s Indifference Property
Introductory Game

- The professor will assign you a number from 1 to the number of people in the class.
- If you are 1 through N/2, your partner will be the individual who is N/2 more than your number, e.g., suppose there are 20 people in the class, the person who is 1 will be paired up with the person who is 11.
You will play rocks-paper-scissor to find out who is player 1 and who is player 2

Player 1 will go to one side of the room, while player 2 will go to the other side of the room

Player 1 will need to decide 10 different times on whether to go Up or Down
  - The person will record each decision on the piece of paper given

Player 2 will need to decide 10 different times on whether to go Left or Right
  - The person will record each decision on the piece of paper given

The payoffs for your decisions are given on the next slide
Introductory Game Cont.

<table>
<thead>
<tr>
<th>Player 1</th>
<th>Player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>P</td>
<td>Q</td>
</tr>
<tr>
<td>Up</td>
<td>1-Q</td>
</tr>
<tr>
<td>Left</td>
<td>Right</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Player 1</th>
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</tr>
</thead>
<tbody>
<tr>
<td>1-P</td>
<td>Down</td>
</tr>
<tr>
<td></td>
<td>4,6</td>
</tr>
<tr>
<td></td>
<td>8,2</td>
</tr>
</tbody>
</table>
Introductory Game Cont.

- Come together with your opponent and figure out what each of your payoffs were for your decisions and write them down on your piece of paper
- Hand in the paper
Discussion

- Was there a best strategy for either player?
- Was there a Nash Equilibrium?
- How did you decide on your decision?
- Would your decision change if you could have conferred after each decision?
There are times in a simultaneous-move game when there is no Nash equilibrium when considering only pure strategies. While these games do not have a Nash with pure strategies, it is possible to find a randomization amongst a set of pure strategies that leads to a Nash.
Defining a Mixed Strategy Cont.

- When you are randomizing your pure strategies, you are in essence choosing a particular strategy to play based on an unsystematic/random way of choosing this strategy.

- When playing a mixed strategy, you get an expected payoff from playing that mixed strategy.
Defining Expected Payoff

- Suppose you have n pure strategies that you can play which will be denoted as $x_n$.
- Suppose you have a set of corresponding probabilities for playing your pure strategies which will be denoted as $p_n$, where the probabilities must all be greater than or equal to zero and they all sum to one.
- Expected payoff can be defined as $p_1 x_1 + p_2 x_2 + \ldots + p_{n-1} x_{n-1} + p_n x_n$. 

$$E(x) = \sum x \cdot p(x)$$
Example of Expected Payoff

- Suppose there are three strategies: Up, Middle, and Down

- Assume the payoff of playing Up is 100, Middle is 200, and Down is 300

- Assume that you will choose Up with probability 0.25, Middle with probability 0.40, and Down with probability 0.35

- The expected payoff would be
  \[0.25 \times 100 + 0.40 \times 200 + 0.35 \times 300 = 25 + 80 + 105 = 210\]
Defining a Nash Equilibrium with Mixed Strategies

- Assume that each player has chosen a set of mixed strategies to play; a Nash equilibrium is said to exist if no player has incentive to change her set of mixed strategies given that the other players do not change their sets of mixed strategies.

- In most cases a Nash equilibrium will exist in mixed strategies even though one does not exist in pure strategies.
Opponent’s Indifference Property

- Under this property, you want to select a set of probabilities for your pure strategies such that it makes the other player indifferent to which strategy she chooses.

- To find this indifference in the case of two strategies, you would select a $p$ for your first strategy and $1-P$ for your second strategy that makes the second player indifferent on which strategy she chooses.
Example of Opponent’s Indifference Property

<table>
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<tr>
<th>Player 1</th>
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<tr>
<td>P</td>
<td>Q</td>
</tr>
<tr>
<td></td>
<td>1-Q</td>
</tr>
<tr>
<td>Up</td>
<td>Left</td>
</tr>
<tr>
<td>70,30</td>
<td>10,90</td>
</tr>
<tr>
<td>Down</td>
<td>Right</td>
</tr>
<tr>
<td>40,60</td>
<td>80,20</td>
</tr>
</tbody>
</table>
Example of Opponent’s Indifference Property Cont.

- Suppose Player 1 wants to choose a P such that Player 2 would be indifferent from choosing Left or Right.
- This would mean that Player 1 wants to set P where 
  \[30P + 60(1-P) = 90P + 20(1-P)\]
- \[30P + 60 - 60P = 90P + 20 - 20P\]
- \[30P - 60P - 90P + 20P = 20 - 60\]
- \[-100P = -40\]
- \[P = 4/10 = 0.4\]
Example of Opponent’s Indifference Property Cont.

- Suppose Player 2 wants to choose a Q such that Player 1 would be indifferent from choosing Up or Down
- This would mean that Player 2 wants to set Q where
  \[70Q + 10 \cdot (1-Q) = 40Q + 80 \cdot (1-Q)\]
  \[70Q + 10 - 10Q = 40Q + 80 - 80Q\]
  \[70Q - 10Q - 40Q + 80P = 80 - 10\]
  \[100Q = 70\]
  \[Q = 7/10 = 0.7\]
Example of Opponent’s Indifference Property Cont.

- The expected payoff of player 1 is:
  \[70 \times 0.7 + 10 \times 0.3 = 52\]
- The expected payoff of player 2 is:
  \[30 \times 0.4 + 60 \times 0.6 = 48\]
Best Response Functions for Each Player

- What is player 1's best response to player 2 playing strategy left with probability Q and playing strategy Right with probability of 1-Q?
- To find this, we multiply player 1's expected payoff for playing each strategy given that she plays up with probability P and Down with probability 1-P by player 2's probability for playing Left (Q) and Right (1-Q)

\[ BR_1(Q) = (70P + 40(1-P))Q + (10P + 80(1-P))(1-Q) \]

- \( BR_1(Q > 0.7) = (P = 1) \)
- \( BR_1(Q = 0.7) = (P = [0, 1]) \)
- \( BR_1(Q < 0.7) = (P = 0) \)
Best Response Functions for Each Player Cont.

- What is player 2’s best response to player 1 playing strategy Up with probability $P$ and playing strategy Down with probability of $1-P$?
- To find this, we multiply player 2’s expected payoff for playing each strategy given that she plays Left with probability $Q$ and Right with probability $1-Q$ by player 2’s probability for playing Up ($P$) and Down ($1-P$)

$$BR_2(P) = (30Q+90(1-Q))P+(60Q+20(1-Q))(1-P)$$

- $BR_2(P>0.4) = (Q=0)$
- $BR_2(P=0.4) = (Q=[0,1])$
- $BR_2(P<0.4) = (Q=1)$
Graph of Both Players’ Best Response Curves

$BR_1(Q)$

$BR_2(P)$
### Mixed Strategies: Assurance Game

<table>
<thead>
<tr>
<th>Player 1</th>
<th>Player 2</th>
<th>Discount Buffalo Wins</th>
<th>Discount Salsa</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-P</td>
<td>Discount Chips</td>
<td>0 , 0</td>
<td>40 , 40</td>
</tr>
<tr>
<td>1-P</td>
<td>Discount Salsa</td>
<td>0 , 0</td>
<td>40 , 40</td>
</tr>
<tr>
<td>P</td>
<td>Discount Beer</td>
<td>20 , 20</td>
<td>0 , 0</td>
</tr>
<tr>
<td>P</td>
<td>Q</td>
<td>1-Q</td>
<td>0 , 0</td>
</tr>
<tr>
<td>P</td>
<td>1-Q</td>
<td>Discount Buffalo Wins</td>
<td>0 , 0</td>
</tr>
</tbody>
</table>
Best Response Functions for Each Player

- What is player 1’s best response to player 2 playing strategy left with probability Q and playing strategy Right with probability of 1-Q?
- To find this, we multiply player 1’s expected payoff for playing each strategy given that she plays up with probability P and Down with probability 1-P by player 2’s probability for playing Left (Q) and Right (1-Q)
  
  \[ BR_1(Q) = (20P + 0(1-P))Q + (0P + 40(1-P))(1-Q) \]

- \[ BR_1(Q > 2/3) = (P = 1) \]
- \[ BR_1(Q = 2/3) = (P = [0,1]) \]
- \[ BR_1(Q < 2/3) = (P = 0) \]
Best Response Functions for Each Player Cont.

- What is player 2’s best response to player 1 playing strategy Up with probability $P$ and playing strategy Down with probability of $1-P$?

- To find this, we multiply player 2’s expected payoff for playing each strategy given that she plays Left with probability $Q$ and Right with probability $1-Q$ by player 2’s probability for playing Up ($P$) and Down ($1-P$)

$$BR_2(P) = (20Q+0(1-Q))P+(0Q+40(1-Q))(1-P)$$

- $BR_2(P>2/3) = (Q=1)$
- $BR_2(P=2/3) = (Q=[0,1])$
- $BR_2(P<2/3) = (Q=0)$
Graph of Both Players’ Best Response Curves
General Results that Come Out of Mixed Strategy

- The equilibrium you get from mixed strategies is weak due to the indifference property.
- There can be counterintuitive results in mixture probabilities; when your payoff increases for a particular pure strategy, you may find yourself using it less.
Mixed Strategies when One Player has Three Strategies

<table>
<thead>
<tr>
<th>Player 1</th>
<th>Player 2</th>
<th>Q</th>
<th>1-Q</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Left</td>
<td>Right</td>
</tr>
<tr>
<td>P1 Up</td>
<td>60, 40</td>
<td></td>
<td>70, 30</td>
</tr>
<tr>
<td>P2 Middle</td>
<td>80, 20</td>
<td></td>
<td>10, 90</td>
</tr>
<tr>
<td>1-P1-P2 Down</td>
<td>75, 25</td>
<td></td>
<td>55, 45</td>
</tr>
</tbody>
</table>
Mixed Strategies When One Player has Three Strategies Cont.

Player 1’s Payoff

- Player 1 plays Up for $0 \leq Q \leq 0.5$
- Player 1 plays Down for $0.5 \leq Q \leq 0.9$
- Player 1 plays Middle for $0.9 \leq Q \leq 1$
Mixed Strategies When Both Players Have Three Strategies

- When both players have three strategies each, you will need to give three probabilities for each player, i.e., $P_1$, $P_2$, and $P_3$ for the first player and $Q_1$, $Q_2$, and $Q_3$ for the second player.

- Since each set of probabilities add to 1, you can define $P_3 = 1 - P_1 - P_2$ and $Q_3 = 1 - Q_1 - Q_2$.

- To solve these problems you will want to find the $P$'s and $Q$'s that will make each player indifferent to the other person’s mixed strategies like was done in the two-by-two game.
Example of 3x3 Strategy Game

<table>
<thead>
<tr>
<th>Player 1</th>
<th>Player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>P1</td>
<td>Q1</td>
</tr>
<tr>
<td>Up</td>
<td>Q2</td>
</tr>
<tr>
<td>9,11</td>
<td>18,2</td>
</tr>
<tr>
<td>18,2</td>
<td>18,2</td>
</tr>
<tr>
<td>P2</td>
<td>Q1-Q2</td>
</tr>
<tr>
<td>Middle</td>
<td>Q2</td>
</tr>
<tr>
<td>17,3</td>
<td>0,20</td>
</tr>
<tr>
<td>17,3</td>
<td>12,8</td>
</tr>
<tr>
<td>1-P1-P2</td>
<td>P1</td>
</tr>
<tr>
<td>Down</td>
<td>1-P2</td>
</tr>
<tr>
<td>19,1</td>
<td>19,1</td>
</tr>
<tr>
<td>19,1</td>
<td>12,8</td>
</tr>
</tbody>
</table>
Final Discussion, Questions, and Thoughts