Terms to Know

- Constraint Boundary, Corner-Point Solutions, Corner-Point Feasible Solution (CPF), Iteration, Iterative Algorithm, Optimality Test, Slack Variables, Augmented Form, Augmented Solution, Basic Solution, Basic Feasible Solution (BF), Non-Basic Variables, Basic Variables, Basis, Initial BF Solution
Terms to Know Cont.

- Minimum Ratio Test, Leaving Basic Variable, Entering Basic Variable, Elementary Algebraic Operations, Simplex Tableau, Pivot Column, Pivot Row, Pivot Number, Elementary Row Operations, Degenerate, Optimal Solution, Artificial-Variable Technique, Artificial Problem, Artificial Variable
Terms to Know Cont.

- Big M Method, Surplus Variable, Two-Phase Method, Shadow Price, Binding Constraints, Sensitive Parameters, Allowable Range, Parametric Linear Programming, Interior Points, Interior-Point Algorithms, Barrier Algorithm, Polynomial Time Algorithm, Exponential Time Algorithm
Wyndor Glass Co. Example Revisited

- \textit{Maximize } Z = 3x_1 + 5x_2 \\
\textit{Subject to:} \\
\begin{align*}
x_1 &\leq 4 \\
2x_2 &\leq 12 \\
3x_1 + 2x_2 &\leq 18 \\
x_1 &\geq 0, \quad x_2 \geq 0
\end{align*}
Graphical View of Wyndor Problem

\[3x_1 + 2x_2 = 18\]

\[2x_2 = 12\]

\[x_1 = 4\]
Items in the Graph to Consider

- Each line is a constraint boundary.
- The intersection of two constraint boundary lines gives a corner-point solution.
  - Corner-point feasible solutions (CPF) are solutions to the intersection of two boundary constraint lines but also meet the criterion of all the other constraints in the model.
    - E.g., (0,0); (0,6); (4,0); (4,3); (2,6)
Items in the Graph to Consider Cont.

- Corner-point infeasible solutions are solutions to two the intersection of two boundary constraint lines but do not meet the criterion of some other constraint in the model.
  - E.g., (0,9); (6,0); (4,6)
- Two CPF solutions are considered adjacent if they share n-1 constraint boundaries where n represents the number of decision variables.
  - E.g., (0,0) is adjacent to (0,6) and (4,0)
  - E.g., (2,6) is adjacent to (4,3) and (0,6)
Optimality Test

- Assume that there is at least one optimal solution.
  - A CPF solution is optimal if there are no other adjacent CPF solutions that increase $Z$. 
Solving the Wyndor Glass Co. Example (Graphical Approach with Simplex Method in Mind)

- **Iteration 0:**
  - Start from an initial CPF solution; usually the origin.
  - Identify adjacent CPF solutions
  - Test to see if you can improve Z by moving to an adjacent CPF solution.
    - If no, you have found the optimal.
    - If yes, move to adjacent CPF solution that improves Z the most and increment to the next iteration.
• **Iteration i:**
  ◦ Start from CPF solution given from iteration i-1.
  ◦ Identify adjacent CPF solutions
  ◦ Test to see if you can improve Z by moving to an adjacent CPF solution.
    • If no, you have found the optimal.
    • If yes, move to adjacent CPF solution that improves Z the most and increment to the next iteration.
Algebraic Approach to the Simplex Method

- To solve a linear programming problem using a computer, a set of algebraic steps are needed.
  - These algebraic steps are needed to allow the computer to solve a set of linear equations.
  - The current Wyndor problem is not set up as a set of linear equations that are met with equality, rather they are expressions using inequality.
Slack Variables

- One way to change an inequality constraint to an equality constraint is to add what is known as a slack variable.
  - The purpose of the slack variable is to take an inequality constraint and turning it into an equality constraint.
  - Suppose we have the Wyndor constraint \( x_1 \leq 4 \), we can define a new variable \( x_3 = 4 - x_1 \) where \( x_3 \geq 0 \).
    - These last two constraints can replace the first constraint.
Creating the Augmented Form of Wyndor’s Model Using Slack Variables

- The following is equivalent to the original Wyndor problem:

Maximize \( Z = 3x_1 + 5x_2 \)

Subject to:

\[
\begin{align*}
  x_1 + x_3 &= 4 \\
  2x_2 + x_4 &= 12 \\
  3x_1 + 2x_2 + x_5 &= 18 \\
  x_1 &\geq 0, \ x_2 \geq 0, \ x_3 \geq 0, \ x_4 \geq 0, \ x_5 \geq 0
\end{align*}
\]
Properties of a Basic Solution that Is Derived from the Augmented Problem

- Each variable in the solution is either a basic variable or a non basic variable
- The number of basic variables should equal the number of constraints
- The non basic variables are set to zero
  - Why?
- The simultaneous solution to the system of equations will give the basic variables
- The basic solution is a basic feasible solution if all the non negativity constraints are satisfied
Solving the Wyndor Problem Using the Simplex Method: Step 0, Write the Set Equations to Solve

\[
\begin{align*}
Z &= -3x_1 - 5x_2 = 0 \\
x_1 + x_3 &= 4 \\
2x_2 + x_4 &= 12 \\
3x_1 + 2x_2 + x_5 &= 18
\end{align*}
\]
Solving the Wyndor Problem Using the Simplex Method: Step 1, Find an Initial Solution

- Because the slack variables were introduced into the problem, a natural initial solution is to set \( x_1 = 0 \) and \( x_2 = 0 \)
  - This implies that \( x_3 = 4, x_4 = 12, \) and \( x_5 = 18 \)
  - This initial solution can be represented as: \((0,0,4,12,18)\)
  - \( x_1 = 0 \) and \( x_2 = 0 \) are the non basic variables for this current setup
    - Why?
  - This implies that \( Z = 0 \)
Solving the Wyndor Problem Using the Simplex Method: Step 2, Test Solution for Optimality

- It is straightforward to see that increasing $x_1$ or $x_2$ would provide a better solution than the current one
  - Why?
Solving the Wyndor Problem Using the Simplex Method: Step 3, Determine Which Variable Should Increase

- Looking at the original equation $Z = 3x_1 + 5x_2$ it appears that it would be best to increase the amount of $x_2$
  - Why?
- $x_2$ becomes known as the entering basic variable
Solving the Wyndor Problem Using the Simplex Method: Step 4, Determine the Amount $x_2$ Should Increase By

- We know that all variables in the problem must be non negative, which implies that:
  - $x_3 = 4 \geq 0$ which implies $x_2 \leq$ Infinity
  - $x_4 = 12 - 2x_2 \geq 0$ which implies $x_2 \leq 6$
  - $x_5 = 18 - 2x_2 \geq 0$ which implies $x_2 \leq 9$

- By the minimum ratio test, $x_2$ is limited to be no larger than 6 which is the largest amount you can increase $x_2$
  - This implies that $x_4 = 0$ and becomes the leaving basic variable
Quick Algebraic Note

- **Rule 1**: You can add one equation to another without affecting the ultimate solution to the set of equations, e.g.:
  - $x_1 = 5$ and $x_2 = 7$ is equivalent to:
    - $x_1 + x_2 = 12$ and $x_2 = 7$
  - The same is true for subtraction

- **Rule 2**: You can divide an equation by a number without affecting the results of the equation, e.g.:
  - $4x_1 + 8x_2 = 12$ is equivalent to:
    - $x_1 + 2x_2 = 3$
Solving the Wyndor Problem Using the Simplex Method: Step 5, Find the New Basic Feasible Solution by Using Elementary Algebraic Operations

- First, divide the row with $x_4$ in it by 2 to get:

$$
\begin{array}{ccc}
Z & -3x_1 & -5x_2 \\
-3x_1 & -5x_2 & = 0 \\
x_1 & +x_3 & = 4 \\
x_2 & +(1/2) x_4 & = 6 \\
3x_1 & +2x_1 & +x_5 = 18
\end{array}
$$
Add 5 times the third row to the first row to get:

\[
\begin{align*}
Z & \quad -3x_1 + (5/2)x_4 = 30 \\
-x_1 + x_3 & = 4 \\
x_2 + (1/2)x_4 & = 6 \\
3x_1 + 2x_2 + x_5 & = 18
\end{align*}
\]
Solving the Wyndor Problem Using the Simplex Method: Step 5, Find the New Basic Feasible Solution by Using Elementary Algebraic Operations Cont.

- Subtract 2 times the third row from the fourth row to get:

\[
\begin{align*}
  Z & -3x_1 + \frac{5}{2}x_4 = 30 \\
  x_1 + x_3 & = 4 \\
  x_2 + \frac{1}{2}x_4 & = 6 \\
  3x_1 - x_4 + x_5 & = 6
\end{align*}
\]
Solving the Wyndor Problem Using the Simplex Method: Step 5, Find the New Basic Feasible Solution by Using Elementary Algebraic Operations Cont.

- The new feasible solution is where \( x_1 = 0 \) and \( x_4 = 0 \)
  - This implies that \( x_2 = 6, x_3 = 4, \) and \( x_5 = 6 \)
  - This gives a new solution of \( (0,6,4,0,6) \)
    - This new solution is adjacent to the previous solution
Solving the Wyndor Problem Using the Simplex Method: Step 6, Test Solution for Optimality

- It is straightforward to see that increasing $x_1$ would provide a better solution than the current one
  - Why?
  - Why not change the $x_4$ variable?
Solving the Wyndor Problem Using the Simplex Method: Step 7, Determine the Amount $x_1$ Should Increase By

- We know that all variables in the problem must be non-negative, which implies that:
  - $x_3 = 4 - x_1 \geq 0$ which implies $x_1 \leq 4$
  - $x_2 = 6 \geq 0$ which implies $x_1 \leq \text{Infinity}$
  - $x_5 = 6 - 3x_1 \geq 0$ which implies $x_1 \leq 2$

- By the minimum ratio test, $x_1$ is limited to be no larger than 2 which is the largest amount you can increase $x_1$
  - This implies that $x_5 = 0$ and becomes the leaving basic variable
Solving the Wyndor Problem Using the Simplex Method: Step 8, Find the New Basic Feasible Solution by Using Elementary Algebraic Operations

- First, divide the row with $x_5$ in it by 3 to get:

\[
\begin{align*}
Z & -3x_1 + \frac{5}{2}x_4 = 30 \\
-x_1 + x_3 & = 4 \\
x_2 + \frac{1}{2}x_4 & = 6 \\
-x_1 - \frac{1}{3}x_4 + \frac{1}{3}x_5 & = 2
\end{align*}
\]
Add 3 times the fourth row to the first row to get:

\[
\begin{align*}
Z &+ \frac{3}{2} x_4 + x_5 = 36 \\
x_1 + x_3 &= 4 \\
x_2 + \frac{1}{2} x_4 &= 6 \\
x_1 - \frac{1}{3}x_4 + \frac{1}{3}x_5 &= 2
\end{align*}
\]
Solving the Wyndor Problem Using the Simplex Method: Step 8, Find the New Basic Feasible Solution by Using Elementary Algebraic Operations Cont.

- Subtract the fourth row from the second row to get:

\[
\begin{align*}
Z & \quad + (3/2) x_4 \quad + x_5 \quad = \quad 36 \\
- (1/3) x_4 \quad & \quad - (1/3) x_5 \quad = \quad 2 \\
(1/2) x_4 \quad & \quad = \quad 6 \\
(1/3) x_4 \quad & \quad + (1/3) x_5 \quad = \quad 2
\end{align*}
\]
The new feasible solution is where $x_4 = 0$ and $x_5 = 0$

- This implies that $x_1 = 2$, $x_2 = 6$, and $x_3 = 6$
- This gives a new solution of $(2,6,2,0,0)$
  - This new solution is adjacent to the previous solution
Solving the Wyndor Problem Using the Simplex Method: Step 9, Test Solution for Optimality

- It is straightforward to see that nothing else would increase $Z$
  - Why?
  - Why not change the $x_4$ or $x_5$ variable?
- The maximum amount of $Z$ is 36 at the optimal solution $(2,6,2,0,0)$
- The problem is done
Class Activity (Not Graded)

- Solve the following problem using the Simplex Method:

  - Maximize $Z = 3x_1 + 4x_2 + 5x_3$

Subject to:

\[
\begin{align*}
3x_1 + 2x_2 + 5x_3 &\leq 150 \\
1x_1 + 4x_2 + 1x_3 &\leq 120 \\
2x_1 + 2x_2 + 2x_3 &\leq 100 \\
x_1 &\geq 0, x_2 \geq 0, x_3 \geq 0
\end{align*}
\]
On Your Own Activity

- Solve the following problem using the Simplex Method:

- Maximize \( Z = x_1 + 2x_2 \)

Subject to:

\[
\begin{align*}
1x_1 + 3x_2 & \leq 8 \\
1x_1 + x_2 & \leq 4 \\
x_1 & \geq 0, \ x_2 \geq 0
\end{align*}
\]
Representing Wyndor in Tabular Form

\[
\begin{align*}
Z &= -3x_1 - 5x_2 = 0 \\
x_1 + x_3 &= 4 \\
2x_2 + x_4 &= 12 \\
3x_1 + 2x_1 + x_5 &= 18
\end{align*}
\]

<table>
<thead>
<tr>
<th>Basic Variable</th>
<th>Equation</th>
<th>Coefficient of:</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Z  x_1  x_2  x_3  x_4  x_5</td>
</tr>
<tr>
<td>Z</td>
<td>0</td>
<td>1  -3  -5  0  0  0</td>
</tr>
<tr>
<td>x_3</td>
<td>1</td>
<td>0  1  0  1  0  0</td>
</tr>
<tr>
<td>x_4</td>
<td>2</td>
<td>0  0  2  0  1  0</td>
</tr>
<tr>
<td>x_5</td>
<td>3</td>
<td>0  3  2  0  0  1</td>
</tr>
</tbody>
</table>
Solving Wyndor Method Using the Tabular Form

• Optimality Test
  ◦ Check to see if any of the coefficients in row one are negative
    • If no, stop because you have the optimal solution
    • If yes, your solution is not optimal and you must go to a first iteration

• An Iteration
  ◦ Find the entering basic variable by selecting the variable, i.e., column, with the largest negative coefficient
    • This column is known as the pivot column
Solving Wyndor Method Using the Tabular Form Cont.

- Next determine the leaving basic variable by applying the minimum ratio test
  - Divide the last column of numbers, i.e., the Right Side column, by the corresponding number in the pivot column
    - If the pivot column has a zero, put infinity in for the number, or a very large number that is several orders of magnitude above the other calculated numbers
  - Select the row with the smallest number after the division
    - This becomes the leaving basic variable
    - This row is known as the pivot row
  - Where the pivot row and pivot column intersect, you will find the pivot number
Solving Wyndor Method Using the Tabular Form Cont.

- Solve for the new basic feasible solution by using elementary row operations to make the pivot number equal to one and all other pivot numbers in the pivot column equal to zero.
- Use the optimality test to test the new BF solution
  - If the new solution is optimal, then stop
  - If the new solution is not optimal do another iteration
In-Class Activity (Not Graded)

- Solve the following problem using the tabular form of the Simplex Method:
  - \textit{Maximize } Z = 3x_1 + 4x_2 + 5x_3
  - \textit{Subject to:}
    - 3x_1 + 2x_2 + 5x_3 \leq 150
    - 1x_1 + 4x_2 + 1x_3 \leq 120
    - 2x_1 + 2x_2 + 2x_3 \leq 100
    - x_1 \geq 0, x_2 \geq 0, x_3 \geq 0
On Your Own Activity

- Solve the following problem using the tabular form of the Simplex Method:

  *Maximize* $Z = x_1 + 2x_2$

*Subject to:*

\[
egin{align*}
1x_1 + 3x_2 & \leq 8 \\
1x_1 + x_2 & \leq 4 \\
x_1 & \geq 0, x_2 \geq 0
\end{align*}
\]
Issues with Simplex Method

• **Tie Between Entering Basic Variables**
  ◦ Break the tie arbitrarily

• **Leaving Basic Variables Tie**
  ◦ Break tie arbitrarily, but it is possible to have problems

• **No Leaving Basic Variable Occurs**
  ◦ This implies you have an unbounded Z

• **Multiple Optimal Solutions**
  ◦ This occurs when the objective function has the same slope as the constraint that the optimal solution(s) are on
Handling Equality Constraints

- Equality constraints can cause problems under the initial solution method because there is no natural starting point for the algorithm
  - To handle this issue, we can use the artificial variable technique and the Big M Method
Revised Wyndor Glass Co. Example with Equality Constraint

- **Maximize** \( Z = 3x_1 + 5x_2 \)

**Subject to:**

- \( x_1 \leq 4 \)
- \( 2x_2 \leq 12 \)
- \( 3x_1 + 2x_2 = 18 \)
- \( x_1 \geq 0, x_2 \geq 0 \)
Creating the Augmented Form of the Revised Wyndor’s Model Using Slack Variables

- The following is equivalent to the original Wyndor problem:
- Maximize $Z = 3x_1 + 5x_2$

Subject to:

\[ x_1 + x_3 = 4 \]
\[ 2x_2 + x_4 = 12 \]
\[ 3x_1 + 2x_2 = 18 \]
\[ x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0 \]
Tabular Form of Revised Wyndor Problem

- Notice in the table below that there is no obvious feasible first solution

<table>
<thead>
<tr>
<th>Basic Variable</th>
<th>Equation</th>
<th>Coefficient of:</th>
<th>Right Side</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Z   x₁   x₂   x₃   x₄</td>
<td></td>
</tr>
<tr>
<td>Z</td>
<td>0</td>
<td>1   -3   -5   0   0</td>
<td>0</td>
</tr>
<tr>
<td>x₃</td>
<td>1</td>
<td>0   1   0   1   0</td>
<td>4</td>
</tr>
<tr>
<td>x₄</td>
<td>2</td>
<td>0   0   2   0   1</td>
<td>12</td>
</tr>
<tr>
<td>x₅</td>
<td>3</td>
<td>0   3   2   0   0</td>
<td>18</td>
</tr>
</tbody>
</table>
The Artificial Variable Techniques and The Big M Method

- This artificial variable technique introduces a non-negative variable similar to the slack variables
  - To introduce this variable, an extremely large penalty must be introduced into the objective function for this variable being positive which will force the variable to be zero in the solution process
    - The value we give to the penalty is known as M which is meant to represent an extremely large number
Creating the Augmented Form of the Revised Wyndor’s Model Using the Artificial Variable Technique

- The following is equivalent to the original Wyndor problem:

Maximize $Z = 3x_1 + 5x_2 - Mx_5$

Subject to:

\[
x_1 + x_3 = 4 \\
2x_2 + x_4 = 12 \\
3x_1 + 2x_2 + x_5 = 18 \\
x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0, x_5 \geq 0
\]
To solve these problems, you need to first get the penalty out of the objective function using elementary row operations. You next follow the simplex method.
The Issue of Negative Right-Hand Sides

• Suppose you have one of your constraints with a negative right-hand side
  ◦ E.g., $2x_1 - 3x_2 \leq -6$

• To handle this issue, you can multiply both sides of the inequality by -1
  ◦ $-1(2x_1 - 3x_2) \leq -1(-6)$ gives
  ◦ $-2x_1 + 3x_2 \geq 6$
  ◦ Notice that the inequality sign reverses
The Issue of ≥ Constraints

- When you have a ≥ constraint instead of ≤ constraint, then you first introduce a surplus variable that acts like a slack variable
  - The surplus variable would have a negative sign in front of it
- You then utilize the artificial variable technique because the surplus variable will change the ≥ constraint to an = constraint
The Issue of Minimization Problems

- Two ways to deal with it
  - Change the instructions in the simplex method
  - Multiply the objective function by \(-1\)
    - Maximizing \(Z\) is the same as minimizing \(-Z\)
    - Why?
Quick Note About the Artificial Variable Technique

- If the original problem has no feasible solution then the final solution will have at least one artificial variable greater than zero
Post-Optimality Analysis

- Re-Optimization
  - For very large problems that may get small changes, it may make sense to start from the previous solution before changes were made to the model

- Shadow Price
  - This value tells you how much the Z will change for small changes in a resource constraint
    - Binding constraints will have positive shadow prices, while non-binding constraints will have a value of zero
    - Shadow prices for the constraints all embedded in the final objective function from the simplex method
Sensitivity Analysis

- Sensitivity analysis is meant to understand how robust your model is to the assumptions made in the model
  - One of the major assumptions made in the model concern the value of the coefficients in the objective function and the constraints
  - Sensitivity analysis can be used to see how much a coefficient can change before the optimal answer changes
Sensitivity Analysis Cont.

- Sensitivity analysis can allow you to put allowable ranges around each of the coefficients
  - An allowable range tells you all the values the coefficient can take before the optimal solution changes